Proving 2^n > n^3 for all n > 9 (updated)

2025-06-26

Prove $2^n > n^3$ for all $n > 9$

Proof:

By mathematical induction.

Let $A(n)$ denote $2^n > n^3$.

For $n = 10$ (base step)

$$2^{10} > 10^3$$
$$1024 > 1000$$

Thus, $A(10)$ is true.

Now, assuming $A(n)$ is true, we get:

$$2^n > n^3$$

Multiplying by 2 on both sides:

$$2^n \cdot 2 > 2n^3$$
$$\implies 2^{n+1} > 2n^3 \qquad \text{(i)}$$

We want to deduce, for $A(n+1)$:

$$2^{n+1} > (n+1)^3$$

To prove this, we have to prove that: (from (i))

$$2n^3 \geq (n+1)^3 \qquad \text{(ii)}$$

$(\forall n > 9)$

If we expand $(n+1)^3$, we get:

$$(n+1)^3 = n^3 + 3n^2 + 3n + 1$$

Now if $(3n^2 + 3n + 1) < n^3$, then $(n+1)^3 < 2n^3$.

If we examine the values for the first several natural numbers, we see that the following is true for all $n \geq 4$, because of the faster rise of the cubic polynomial.

$$f(x) = x^3$$
$$g(x) = 3x^2 + 3x + 1$$

x	$f(x)$	$g(x)$
1	1	7
2	8	19
3	27	37
4	$\boxed{64}$	61
5	125	91
6	216	127
7	343	169
...	...	...
10	1000	331

Clearly, this is true for all $n > 9$.

Thus, $2n^3 > n^3 > (n+1)^3$

And which follows that (i) must be true.

Consequently:

$$2^{n+1} > (n+1)^3$$

Which follows that $A(n)$ must be true.

Therefore, by the principle of mathematical induction, we have proven that $2^n > n^3$ for all $n > 9$.